Lecture 10 02/19/2018

Review of Magnetostatics (Cont'd)

We note that the magnetic dipole moment of a localized current distribution does not depend on the choice of origin:

(localized current)

$$\vec{x} \to \vec{x} + \vec{R} \Rightarrow \vec{m} \to \vec{m} + \frac{1}{2}\int \vec{R} \times \vec{J}(\vec{x}')\, d\sigma' = \vec{m} + \frac{1}{2}\vec{R} \times \cancelto{0}{\int \vec{J}(\vec{x}')\, d\sigma'} = \vec{m}$$

This is consistent with the absence of magnetic monopoles; and hence magnetic dipole being the first non-vanishing multipole, as we saw in the case of electrostatics.

Considering N point charges $q_1, \ldots, q_N$ moving with respective velocities $\vec{v}_1, \ldots, \vec{v}_N$, we have:

$$\vec{m} = \frac{1}{2}\sum_{i=1}^{N} \vec{x}_i \times q\vec{v}_i = \sum_{i=1}^{N} \frac{q_i}{2m_i}(\vec{x}_i \times m_i \vec{v}_i) = \sum_{i=1}^{N} \frac{q_i}{2m_i}\vec{L}_i$$

Including the spin angular momentum, we have:

$$\vec{m} = \sum_{i=1}^{N} \frac{q_i}{2m_i}\vec{L}_i + \sum_{i=1}^{N}\frac{q_i}{2m_i} g_s \vec{S}_i = \boxed{\sum_{i=1}^{N}\frac{q_i}{2m_i}(\vec{L}_i + \underbrace{g_s}_{g_s \simeq 2}\vec{S}_i)}$$

Now, let us consider a localized current distribution in an external magnetic field $\vec{B}_{ext}(\vec{x})$. Expanding the external field as a Taylor series about an origin chosen within the current volume, we have:

$$\vec{B}_{ext}(\vec{x}) = \vec{B}_{ext}(0) + (\vec{x}\cdot\vec{\nabla})\vec{B}_{ext}(0) + \cdots$$

The force on the current distribution is:

$$\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d\tau = \int \vec{J}(\vec{x}) \times \vec{B}_{ext}(0)\, d\tau + \int \vec{J}(\vec{x}) \times (\vec{x}\cdot\vec{\nabla})\vec{B}_{ext}(0)\, d\tau + \cdots = \cancelto{0}{\int \vec{J}(\vec{x})\, d\tau} \times \vec{B}_{ext}(0) + \int \vec{J}(\vec{x}) \times (\vec{x}\cdot\vec{\nabla}\, \vec{B}_{ext}(0))\, d\tau + \cdots$$

Writing $\vec{F} = F_i \hat{e}_i$, we find (to the first non-vanishing order):

$$F_i = \varepsilon_{ijk} \int J_j(\vec{x})\, x_l\, \partial_l B_{ext,k}(0)\, d\tau = \varepsilon_{ijk} (\vec{m} \times \vec{\nabla})_j B_{ext,k}(0)$$

Thus:

$$\vec{F} = (\vec{m} \times \vec{\nabla}) \times \vec{B}_{ext}(0) + \cdots = \vec{\nabla}(\vec{m}\cdot\vec{B}_{ext}(0)) - \vec{m}\,\cancelto{0}{(\vec{\nabla}\cdot\vec{B}_{ext}(0))}$$

$$\Rightarrow \boxed{\vec{F} = -\vec{\nabla} U, \quad U = -\vec{m}\cdot\vec{B}_{ext}} \quad *$$

U is the potential energy due to interaction of a magnetic dipole with an external magnetic field. The minimum energy is attained when the magnetic dipole aligns with the magnetic field.

The torque on the localized distribution follows:

$$\vec{N} = \int \vec{x} \times (\vec{J}(\vec{x}) \times \vec{B}_{ext}(\vec{x}))\, d\tau = \int \vec{x} \times (\vec{J}(\vec{x}) \times \vec{B}_{ext}(0))\, d\tau + \cdots$$

$$= \int (\vec{x} \cdot \vec{B}_{ext}(0))\, \vec{J}(\vec{x})\, d\tau - \vec{B}_{ext}(0) \int \vec{x} \cdot \vec{J}(\vec{x})\, d\tau + \cdots$$

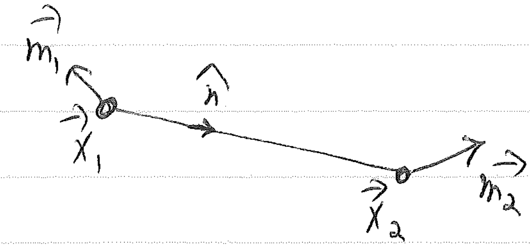
It can be shown that for a localized current distribution $\int x_i J_i \, d\tau = 0$. Also, the first term on the right-hand side may be transformed in a similar way as we did for finding $\vec{F}$. Hence:

$$\vec{N} = \vec{m} \times \vec{B}_{ext}(0) + \cdots$$

Note that this is consistent with $\vec{m} \parallel \vec{B}_{ext}(0)$ for the minimum energy configuration, as $\vec{N} = 0$ is expected in equilibrium.

Couple of remarks are in order. First, the interaction energy between two magnetic dipoles $\vec{m}_1$ and $\vec{m}_2$ can be found by substituting the magnetic field due to one of the dipoles in the expression * on the previous page:

$$U = -\vec{m}_2 \cdot \vec{B}_1(\vec{x}_2) = \frac{\mu_0}{4\pi|\vec{x}_1 - \vec{x}_2|^3}\left[\vec{m}_1 \cdot \vec{m}_2 - 3(\vec{m}_1 \cdot \hat{n})(\vec{m}_2 \cdot \hat{n})\right]$$



$$= -\vec{m}_1 \cdot \vec{B}_2(\vec{x}_1)$$

Second, for a localized current distribution, we have:

$$\int_V B(\vec{x})\, d\sigma = \frac{2}{3}\mu_0 \vec{m}$$

Here, V is a spherical volume that contains the entire current distribution, and $\vec{m}$ is the magnetic dipole moment of the distribution.

Magnetic Materials

In addition to free currents, that are carried by free charges, we also have magnetization currents carried by bound charges in the constituent atoms/molecules (or by spins, which are intrinsically quantum mechanical).

The "magnetization" $\vec{M}(\vec{x})$ inside the material is defined as $\lim\limits_{\Delta V \to 0} \frac{\Delta \vec{m}}{\Delta V}$, which is basically the magnetic dipole density.

The vector potential due to the two currents is:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d\tau' + \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{x}') \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\,d\tau' =$$

$$\frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d\tau' + \frac{\mu_0}{4\pi}\int \vec{M}(\vec{x}') \times \vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d\tau'$$

But:

$$\vec{M}(\vec{x}') \times \vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -\vec{\nabla}' \times \frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

After using the identity $\int_V \vec{\nabla} \times \vec{v}\, d\tau = \oint_S (\hat{n} \times \vec{v})\, da$, we find:

(for localized current)

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{x}') + \vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d\tau' + \frac{\mu_0}{4\pi}\oint_S \cancelto{0}{\frac{\vec{M}(\vec{x}') \times \hat{n}'}{|\vec{x}-\vec{x}'|}}\,da'$$

Therefore:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}') + \vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d\tau'$$

Now:

(as shown before)

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\cancelto{0}{\vec{\nabla} \cdot \vec{A}}) - \nabla^2 \vec{A} = \mu_0\left(\vec{J}(\vec{x}) + \right.$$

$$\left.\vec{\nabla} \times \vec{M}(\vec{x})\right) \Rightarrow \vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{J} \Rightarrow \boxed{\vec{\nabla} \times \vec{H} = \vec{J}, \quad \left(\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}\right)}$$

Henceforth, we call the $\vec{B}$ field the "magnetic field of induction";

or simply the "magnetic induction", and the $\vec{H}$ field the "magnetic field intensity", or simply the "magnetic field".

It is worthwhile to underline that the fundamental field is the $\vec{B}$ field, just like the $\vec{E}$ field in the case of electrostatics. The $\vec{H}$ field, similar to the $\vec{D}$ field in electrostatics, is a derived field that is introduced as a matter of convenience to take into account of the currents due to bound atomic/molecular charges.